

Global aspects of polarization optics and coset space geometry

Arvind*

*Department of Physical Sciences, Indian Institute of Science Education & Research (IISER) Mohali,
Sector 81 SAS Nagar, Manauli PO 140306 Punjab India.*

S. Chaturvedi†

*Department of Physics, Indian Institute of Science Education & Research (IISER) Bhopal,
Bhopal Bypass Road, Bhauri, Bhopal 462066 India*

N. Mukunda‡

*INSA C V Raman Research Professor, Indian Academy of Sciences,
C V Raman Avenue, Sadashivanagar, Bangalore 560080 India*

We use group theoretic ideas and coset space methods to deal with problems in polarization optics of a global nature. These include the possibility of a globally smooth phase convention for electric fields for all points on the Poincaré sphere, and a similar possibility of real or complex bases of transverse electric vectors for all possible propagation directions. It is shown that these methods help in understanding some known results in an effective manner, and in answering new questions as well. We find that apart from the groups $SU(2)$ and $SO(3)$ which occur naturally in these problems, the group $SU(3)$ also plays an important role.

I. INTRODUCTION

It is well known that the Poincaré sphere [1] representation of the pure polarization states of a plane electromagnetic wave (with fixed frequency and propagation direction) is intimately related to the properties of the two-dimensional unitary unimodular group $SU(2)$ [2]. Similarly if one considers choices of transverse real electric field vectors corresponding to all possible propagation directions in space, the real rotation group $SO(3)$ becomes relevant. In both cases, questions of a global nature can be analysed particularly effectively if they are cast into a group theoretical form.

In this paper such an approach is used to analyse three such questions. The first is the search for a globally smooth phase convention for transverse electric field vectors corresponding to all points on the Poincaré sphere [3]. This question is shown to have a simple and elegant resolution when examined in the framework of the group $SU(2)$. The second and third questions are concerned with choices of globally smooth basis states for transverse electric field vectors as the propagation direction varies over all points on the sphere of directions in physical space. The second question deals with the case of real fields, while the third question asks whether such a smooth basis is possible for complex electric fields. Here it is a well known result that if we limit ourselves to real fields (corresponding to linear polarizations), such bases do not exist because of an obstruction [4].

The relevant mathematical result is that the sphere S^2 is not parallelizable, more informally that it ‘cannot be

combed’. We will see that relating this question to $SO(3)$ leads to an immediate and illuminating understanding of this result. However it has been pointed out recently that if one goes to the complex domain and allows electric field vectors in all possible (pure) polarization states, the obstruction disappears [5]. There exist bases of complex electric fields which are globally smooth over the sphere of directions. Here too it turns out that the use of the group $SU(2)$, and to a limited extent of $SU(3)$ as well, clarifies the situation greatly [6]. We develop in full detail a simple and attractive group theory based choice of a complex globally smooth basis over the sphere of directions, by combining the answers to the first two questions in such a way that the obstructions seen in them compensate, or annihilate, one another. We also describe an important property of the set of all such globally smooth choices.

The material of this paper is organized as follows. In Section II we address the first of the three questions mentioned above and recast it in the language of group theory. In particular, we show that the problem of finding a globally smooth phase convention over the Poincaré sphere is entirely equivalent to the problem of choosing representatives of the coset space $SU(2)/U(1)$ in a manner that is smooth over all points on the Poincaré sphere. This reformulation of the original question not only permits us to furnish a simple proof of the well known impossibility of a globally smooth phase convention over the Poincaré sphere but, on the positive side, enables us to provide a parameterization that works at all but one point (the South pole) of the Poincaré sphere. In Section III, restricting ourselves to real electric fields, we turn to the question of finding a real orthonormal basis in the two dimensional tangent plane at each point on the sphere of directions in a smooth manner. Here, in the same spirit as in Section II, we show that this exercise

* arvind@iisermohali.ac.in

† subhash@iiserbhopal.ac.in

‡ nmukunda@gmail.com

reduces to finding a smooth choice of representatives, this time for the coset space $SO(3)/SO(2)$. Again, as before, besides providing a simple proof of the well known non existence of such bases, the use of group theoretic language facilitates construction of bases with the desired properties at all but one point on the sphere of directions. Next in Section III we reexamine this question allowing the electric field to be complex. In other words we ask the question whether or not it is possible to come up with a smooth choice of orthonormal bases in the complexified tangent spaces at each point on the sphere of directions. The complexification brings the group $SU(3)$ into play in a natural way and we show that by a judicious use of the constructs developed earlier, one is not only able to answer the question at hand in the affirmative but is also able to give an elegant and essentially unique construction of the desired bases. Section IV contains our concluding remarks and further outlook.

II. GLOBALLY SMOOTH PHASE CONVENTION OVER THE POINCARÉ SPHERE– THE OBSTRUCTION

Consider plane electromagnetic waves (with fixed frequency ω and propagating along the positive z -axis) in various states of pure polarization. Denote the two-component complex electric field vector in the transverse $x - y$ plane, say at $z = 0$, by

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \in \mathbb{C}^2. \quad (1)$$

The intensity in suitable units is $I = \mathbf{E}^\dagger \mathbf{E}$, and for simplicity it will hereafter be set equal to unity. The polarization state is represented by a point on the Poincaré sphere $\mathbb{S}_{\text{pol}}^2$, given in terms of \mathbf{E} by

$$\hat{\mathbf{n}} = \mathbf{E}^\dagger \boldsymbol{\tau} \mathbf{E} \in \mathbb{S}_{\text{pol}}^2. \quad (2)$$

Here we follow the polarization optics conventions for $\boldsymbol{\tau}$ matrices:

$$\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (3)$$

which are related by a cyclic permutation to the Pauli matrices $\boldsymbol{\sigma}$ normally used in quantum mechanics: $\tau_1 = \sigma_3, \tau_2 = \sigma_1, \tau_3 = \sigma_2$. The three-vector $\hat{\mathbf{n}}$ is real, has unit length, and is unchanged by an overall phase transformation $\mathbf{E} \rightarrow \mathbf{E}' = e^{i\phi} \mathbf{E}$. Points on the equator $n_3 = 0$ (\mathbf{E} real apart from an overall phase) correspond to linear polarizations. The two poles $(0, 0, \pm 1)$ represent circular polarizations, RCP or LCP, for $\mathbf{E} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ apart, again, from overall phases. A physically interesting question is this: for each $\hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2$, is it possible to choose $\mathbf{E}(\hat{\mathbf{n}}) \in \mathbb{C}^2$ such that

$$\mathbf{E}(\hat{\mathbf{n}})^\dagger \boldsymbol{\tau} \mathbf{E}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}, \quad (4)$$

with $\mathbf{E}(\hat{\mathbf{n}})$ varying smoothly with respect to $\hat{\mathbf{n}}$ for all $\hat{\mathbf{n}}$? We can recast this question in terms of the group $SU(2)$ as follows. There is a simple one-to-one correspondence between $SU(2)$ elements and normalized two-component complex column vectors (like \mathbf{E} above):

$$\begin{aligned} u &\in SU(2) \Leftrightarrow \\ u &= \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}, |\alpha|^2 + |\beta|^2 = 1 \Leftrightarrow \\ \boldsymbol{\xi} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2, \quad \boldsymbol{\xi}^\dagger \boldsymbol{\xi} = 1; \\ u(\boldsymbol{\xi}) &= (\boldsymbol{\xi}, -i\tau_3 \boldsymbol{\xi}^*). \end{aligned} \quad (5)$$

This kind of relationship is special to $SU(2)$ [7, 8], and it shows immediately that, since $\boldsymbol{\xi} \in \mathbb{S}^3$, as a manifold

$$SU(2) \sim \mathbb{S}^3. \quad (6)$$

Moreover, $u(\boldsymbol{\xi})$ obeys a ‘covariance condition’

$$Su(\boldsymbol{\xi}) = u(S\boldsymbol{\xi}), \quad \text{any } S \in SU(2), \quad (7)$$

which is very useful.

With u and $\boldsymbol{\xi}$ related as in Eqn. (5), we find that

$$\begin{aligned} u\tau_1 u^{-1} &= \hat{\mathbf{n}} \cdot \boldsymbol{\tau}, \\ \hat{\mathbf{n}} &= \boldsymbol{\xi}^\dagger \boldsymbol{\tau} \boldsymbol{\xi} = (|\alpha|^2 - |\beta|^2, 2\Re(\alpha^* \beta), 2\Im(\alpha^* \beta)); \\ u' &= ue^{i\phi\tau_1} \Leftrightarrow \boldsymbol{\xi}' = e^{i\phi} \boldsymbol{\xi}. \end{aligned} \quad (8)$$

The set of elements $\{e^{i\phi\tau_1}, 0 \leq \phi < 2\pi\} \subset SU(2)$ defines a $U(1)$ subgroup, leading to the coset space $SU(2)/U(1)$ which can be identified with $\mathbb{S}_{\text{pol}}^2$. We see that $\hat{\mathbf{n}}$ in Eqn. (8) is determined by $\boldsymbol{\xi}$ in exactly the same way as Eqn. (2) determines the polarization state of \mathbf{E} . In the present case we view Eqn. (8) as defining the projection map $\pi : SU(2) \rightarrow SU(2)/U(1) = \mathbb{S}_{\text{pol}}^2$:

$$\begin{aligned} u \in SU(2) \rightarrow \pi(u) &= \hat{\mathbf{n}} = \boldsymbol{\xi}^\dagger \boldsymbol{\tau} \boldsymbol{\xi} \in \mathbb{S}_{\text{pol}}^2, \\ \pi(ue^{i\phi\tau_1}) &= \pi(u). \end{aligned} \quad (9)$$

Conversely, each $\hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2$ determines a corresponding coset in $SU(2)$:

$$\begin{aligned} \hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2 &\Leftrightarrow \\ \mathcal{C}(\hat{\mathbf{n}}) &= \{u \in SU(2) | \pi(u) = \hat{\mathbf{n}}\} = \pi^{-1}(\hat{\mathbf{n}}) \subset SU(2) \end{aligned} \quad (10)$$

The search for a smooth $\mathbf{E}(\hat{\mathbf{n}})$ for all $\hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2$ obeying Eqn. (4) is the same as the search for a smooth $\boldsymbol{\xi}(\hat{\mathbf{n}})$ for all $\hat{\mathbf{n}}$ such that

$$\boldsymbol{\xi}(\hat{\mathbf{n}})^\dagger \boldsymbol{\tau} \boldsymbol{\xi}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}. \quad (11)$$

Since by Eqn. (5) u and $\boldsymbol{\xi}$ determine one another, this in turn is the same as the search for a smooth coset representative $u(\hat{\mathbf{n}}) \in \mathcal{C}(\hat{\mathbf{n}})$ for all $\hat{\mathbf{n}}$.

However such globally smooth coset representatives are not possible. Suppose one such, $u_0(\hat{\mathbf{n}})$ say, did exist. It would then permit the expression of any $u \in SU(2)$ in a

globally smooth way as a product of a coset representative and a subgroup element:

$$\begin{aligned} u &\in SU(2), \quad \pi(u) = \hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2 : \\ u &= u_0(\hat{\mathbf{n}})e^{i\phi\tau_1}, \quad 0 \leq \phi < 2\pi. \end{aligned} \quad (12)$$

This would imply that as a manifold $SU(2) \simeq \mathbb{S}^3 \simeq \mathbb{S}^2 \times U(1) = \mathbb{S}^2 \times \mathbb{S}^1$, which is easily seen to be false. Therefore no globally smooth $\xi(\hat{\mathbf{n}})$ obeying Eqn. (11), or $\mathbf{E}(\hat{\mathbf{n}})$ obeying Eqn. (4), can be found.

While it is the case that no smooth coset representative $u_0(\hat{\mathbf{n}})$ for all $\hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2$ is available, we can define such a coset representative for all $\hat{\mathbf{n}}$ except, for example, at the South pole $(0, 0, -1)$ of $\mathbb{S}_{\text{pol}}^2$. Here is a simple construction, with $\hat{\mathbf{n}}$ parametrised by spherical polar angles θ, ϕ :

$$\begin{aligned} \hat{\mathbf{n}} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in \mathbb{S}_{\text{pol}}^2, \\ 0 &\leq \theta < \pi, 0 \leq \phi < 2\pi; \\ u_0(\hat{\mathbf{n}}) &= (\xi_0(\hat{\mathbf{n}}), -i\tau_3\xi_0(\hat{\mathbf{n}})^*), \\ \xi_0(\hat{\mathbf{n}}) &= S_0 \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix}, \end{aligned}$$

$$S_0 = \frac{1}{2}(\mathbb{1} + i\tau_1 + i\tau_2 + i\tau_3) = \frac{1}{2} \begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \in SU(2) \quad (13)$$

where the matrix S_0 connects the $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ matrices:

$$\tau_j = S_0 \sigma_j S_0^{-1}, j = 1, 2, 3. \quad (14)$$

One can check that Eqns.(8) are obeyed for all $\hat{\mathbf{n}}$ except the South pole. At the North pole, we do find that these expressions are all well-defined:

$$\begin{aligned} \theta \rightarrow 0 : \quad \xi_0(\theta, \phi) &\rightarrow \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \\ u_0(\theta, \phi) &\rightarrow S_0. \end{aligned} \quad (15)$$

On the other hand, at the South pole the limiting expressions are ϕ dependent, hence undefined:

$$\begin{aligned} \theta \rightarrow \pi : \quad \xi_0(\theta, \phi) &\rightarrow \frac{e^{i(\pi/4+\phi)}}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \\ u_0(\theta, \phi) &\rightarrow -iS_0\tau_3e^{i\phi\tau_1}. \end{aligned} \quad (16)$$

Nevertheless, as we shall see in the next Section, this construction is useful in another context.

III. BASES OF ELECTRIC FIELD VECTORS FOR ALL PROPAGATION DIRECTIONS

We turn next to the consideration of some global problems connected with the collection of all plane waves (with fixed frequency ω) with all possible propagation directions. We denote the sphere of propagation directions in physical space by $\mathbb{S}_{\text{dir}}^2$, to be distinguished from the Poincaré sphere $\mathbb{S}_{\text{pol}}^2$. For any unit vector $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$, the

complex three-component electric field \mathbf{E} (omitting the plane wave factor $e^{-i\omega(t-\hat{\mathbf{k}}\cdot\mathbf{x}/c)}$) obeys the transversality condition

$$\hat{\mathbf{k}} \cdot \mathbf{E} = 0, \quad (17)$$

so it is essentially two-dimensional.

To begin with, consider real fields. Then the vector \mathbf{E} lies in the two-dimensional plane in physical space tangent to $\mathbb{S}_{\text{dir}}^2$ at $\hat{\mathbf{k}}$:

$$\mathbf{E} \text{ real}, \quad \hat{\mathbf{k}} \cdot \mathbf{E} = 0 \Leftrightarrow \mathbf{E} \in T_{\hat{\mathbf{k}}} \mathbb{S}_{\text{dir}}^2 \subset \mathbb{R}^3. \quad (18)$$

We now ask whether it is possible to choose an orthonormal basis of real vectors $\mathbf{E}^{(a)}(\hat{\mathbf{k}})$ for each $\hat{\mathbf{k}}$ obeying

$$\begin{aligned} \hat{\mathbf{k}} \cdot \mathbf{E}^{(a)}(\hat{\mathbf{k}}) &= 0, \quad \mathbf{E}^{(a)}(\hat{\mathbf{k}}) \cdot \mathbf{E}^{(b)}(\hat{\mathbf{k}}) = \delta_{ab}, \\ \mathbf{E}^{(1)}(\hat{\mathbf{k}}) \wedge \mathbf{E}^{(2)}(\hat{\mathbf{k}}) &= \hat{\mathbf{k}}, \quad a, b = 1, 2, \end{aligned} \quad (19)$$

in a globally smooth way. We have included here the condition that $(\mathbf{E}^{(1)}(\hat{\mathbf{k}}), \mathbf{E}^{(2)}(\hat{\mathbf{k}}), \hat{\mathbf{k}})$ form a right handed system; they obviously form an orthonormal system in three dimensions. (Actually, it suffices to be able to choose one real vector $\mathbf{E}^{(1)}(\hat{\mathbf{k}})$ for each $\hat{\mathbf{k}}$ obeying

$$\hat{\mathbf{k}} \cdot \mathbf{E}^{(1)}(\hat{\mathbf{k}}) = 0, \quad \mathbf{E}^{(1)}(\hat{\mathbf{k}}) \cdot \mathbf{E}^{(1)}(\hat{\mathbf{k}}) = 1. \quad (20)$$

Then if we define

$$\mathbf{E}^{(2)}(\hat{\mathbf{k}}) = \hat{\mathbf{k}} \wedge \mathbf{E}^{(1)}(\hat{\mathbf{k}}), \quad (21)$$

all of Eqns. (19) are obeyed). If such a choice were possible, for each $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$ we could define an element $R(\hat{\mathbf{k}}) \in SO(3)$, the proper real orthogonal rotation group in three dimensions, carrying \mathbf{e}_3 to $\hat{\mathbf{k}}$:

$$\begin{aligned} \hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2 &\rightarrow R(\hat{\mathbf{k}}) \in SO(3) : \\ R(\hat{\mathbf{k}})(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) &= (\mathbf{E}^{(1)}(\hat{\mathbf{k}}), \mathbf{E}^{(2)}(\hat{\mathbf{k}}), \hat{\mathbf{k}}). \end{aligned} \quad (22)$$

Here $\mathbf{e}_j, j = 1, 2, 3$ are the unit vectors along the three Cartesian coordinate axes in physical space. Therefore the columns of the matrix $R(\hat{\mathbf{k}})$ are:

$$\begin{aligned} R_{j1}(\hat{\mathbf{k}}) &= \mathbf{E}_j^{(1)}(\hat{\mathbf{k}}), \quad R_{j2}(\hat{\mathbf{k}}) = \mathbf{E}_j^{(2)}(\hat{\mathbf{k}}), \quad R_{j3}(\hat{\mathbf{k}}) = \hat{k}_j, \\ j &= 1, 2, 3. \end{aligned} \quad (23)$$

The elements of $SO(3)$ leaving \mathbf{e}_3 invariant are rotations in the xy plane, forming an $SO(2)$ subgroup of $SO(3)$:

$$\begin{aligned} SO(2) &= \left\{ R_3(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid 0 \leq \phi < 2\pi \right\} \subset SO(3), \\ R_3(\phi)\mathbf{e}_3 &= \mathbf{e}_3. \end{aligned} \quad (24)$$

Similar to the situation with $SU(2)$, here too we have the coset space identification

$$SO(3)/SO(2) \simeq \mathbb{S}_{\text{dir}}^2. \quad (25)$$

Therefore if a choice of $R(\hat{\mathbf{k}})$ smooth over all $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$ did exist, it would be a coset representative and we could use it to express every $R \in SO(3)$ smoothly as a product,

$$\begin{aligned} R \in SO(3), \quad R\mathbf{e}_3 = \hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2 : \\ R = R(\hat{\mathbf{k}})R_3(\phi), \quad 0 \leq \phi < 2\pi, \end{aligned} \quad (26)$$

similar to Eqn. (12). However this would mean that $SO(3)$ has the global structure of $\mathbb{S}^2 \times SO(2)$. This conflicts with the fact that it is \mathbb{S}^3/\sim , where \sim is the identification of antipodal points on \mathbb{S}^3 . Therefore globally smooth choices of $R(\hat{\mathbf{k}}) \in SO(3)$ obeying Eqn. (23) are not possible.

As mentioned earlier, this is a well-known result [9]. Another even more elementary analytic proof is given later.

Now we extend this analysis by considering complex electric field vectors. This means that at each $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$, we go from the real two-dimensional tangent plane $T_{\hat{\mathbf{k}}} \mathbb{S}_{\text{dir}}^2 \subset \mathbb{R}^3$ to its complexification which is no longer contained in \mathbb{R}^3 :

$$\begin{aligned} \hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2 : \quad T_{\hat{\mathbf{k}}} \mathbb{S}_{\text{dir}}^2 \rightarrow (T_{\hat{\mathbf{k}}} \mathbb{S}_{\text{dir}}^2)^c = \\ \{\mathbf{E} = \text{complex three-dimensional vector} \mid \hat{\mathbf{k}} \cdot \mathbf{E} = 0\}. \end{aligned} \quad (27)$$

It now turns out that it is possible to find (in infinitely many ways) orthonormal bases for these complexified tangent spaces, which are globally smooth with respect to $\hat{\mathbf{k}}$ [4]. The complexification in Eqn. (27) suggests that it is useful to extend the groups $SU(2)$ and $SO(3)$ so far used to $SU(3)$, the group of complex unitary unimodular matrices in three dimensions. This contains both $SU(2)$ and $SO(3)$ as subgroups. To begin with, let us write \mathcal{A} for a general matrix in the unitary group $U(3)$. If we demand that the third column \mathcal{A}_{j3} be the components of a chosen $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$, unitarity of \mathcal{A} guarantees that the first two columns of \mathcal{A} form an orthonormal basis for $(T_{\hat{\mathbf{k}}} \mathbb{S}_{\text{dir}}^2)^c$:

$$\begin{aligned} \mathcal{A}_{ja} = E_j^{(a)}, \quad \mathcal{A}_{j3} = \hat{k}_j : \\ \mathcal{A}^\dagger \mathcal{A} = \mathbb{1}_{3 \times 3} \Leftrightarrow E_j^{(a)*} E_j^{(b)} = \delta_{ab}, \quad \hat{k}_j E_j^{(a)} = 0, \quad a, b = 1, 2. \end{aligned} \quad (28)$$

We then find easily that \mathcal{A} has a rather simple form:

$$\begin{aligned} \mathcal{A} = R \begin{pmatrix} u & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R \in SO(3), u \in U(2), \\ R_{j3} = \hat{k}_j. \end{aligned} \quad (29)$$

As in Eq. (19) let us now add the righthandedness condition in the sense

$$\varepsilon_{jkl} \mathcal{A}_{k1} \mathcal{A}_{l2} = \mathcal{A}_{j3} = \hat{k}_j. \quad (30)$$

This implies

$$\varepsilon_{jkl} \mathcal{A}_{k1} \mathcal{A}_{l2} \mathcal{A}_{j3} = \det \mathcal{A} = 1, \quad (31)$$

that is, $\mathcal{A} \in SU(3)$. Then we find that in the structure (29) for \mathcal{A} we must have $u \in SU(2)$. This will hereafter be assumed. Such matrices \mathcal{A} form a subset, not a subgroup, in $SU(3)$. The breakup (29) of \mathcal{A} into two factors is however not unique since there are shared elements:

$$R_3(\phi) = \begin{pmatrix} e^{-i\phi\tau_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (32)$$

Let us parametrise $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$ in the same way as $\hat{\mathbf{n}} \in \mathbb{S}_{\text{pol}}^2$ in Eqn. (13):

$$\begin{aligned} \hat{\mathbf{k}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in \mathbb{S}_{\text{dir}}^2, \\ 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi. \end{aligned} \quad (33)$$

Momentarily avoiding the South pole $\theta = \pi$, let us define $R_0(\theta, \phi) \in SO(3)$ as

$$\begin{aligned} R_0(\theta, \phi) = R_3(\phi)R_2(\theta)R_3(\phi)^{-1}, \\ R_2(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \in SO(3), \\ R_0(\theta, \phi)\mathbf{e}_3 = \hat{\mathbf{k}}. \end{aligned} \quad (34)$$

At the North pole $R_0(\theta, \phi)$ is obviously well-defined: $R_0(0, \phi) = \mathbb{1}$. However as we approach the South pole we find a multivaluedness:

$$\begin{aligned} \theta \rightarrow \pi : R_0(\theta, \phi) \rightarrow \begin{pmatrix} -\cos 2\phi & -\sin 2\phi & 0 \\ -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} -\tau_1 e^{2i\phi\tau_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (35)$$

Let us then write

$$\begin{aligned} \mathcal{A}(\theta, \phi) = R_0(\theta, \phi) \begin{pmatrix} u(\theta, \phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SU(3), \\ u(\theta, \phi) \in SU(2), \quad \mathcal{A}(\theta, \phi)\mathbf{e}_3 = \hat{\mathbf{k}}. \end{aligned} \quad (36)$$

The conditions for $\mathcal{A}(\theta, \phi)$ to be well-defined and globally smooth over $\mathbb{S}_{\text{dir}}^2$ are, apart from smooth dependences on θ and ϕ :

$$\begin{aligned} 0 < \theta < \pi : \mathcal{A}(\theta, \phi + 2\pi) &= \mathcal{A}(\theta, \phi); \\ \theta \rightarrow 0 : \mathcal{A}(\theta, \phi) &\rightarrow \begin{pmatrix} u(0, \phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \phi\text{-independent}; \\ \theta \rightarrow \pi : \mathcal{A}(\theta, \phi) &\rightarrow \begin{pmatrix} -\tau_1 e^{2i\phi\tau_3} u(\pi, \phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \phi\text{-independent}. \end{aligned} \quad (37)$$

These translate into conditions on $u(\theta, \phi)$:

$$\begin{aligned} 0 < \theta < \pi & : u(\theta, \phi + 2\pi) = u(\theta, \phi); \\ \theta \rightarrow 0 & : u(0, \phi) = u_0 \in SU(2); \\ \theta \rightarrow \pi & : u(\pi, \phi) = e^{-2i\phi\tau_3} u_\pi, u_\pi \in SU(2). \end{aligned} \quad (38)$$

We will develop an expression for $u(\theta, \phi)$ using the results of Section II, but let us now show by a simple argument the existence of an obstruction in the real domain. Such a possibility would correspond to $u(\theta, \phi) = e^{-i\alpha(\theta, \phi)\tau_3} \in SO(2) \subset SU(2)$. Then the conditions (38) require that $\alpha(\theta, \phi)$ obey, apart from smoothness in θ and ϕ :

$$\begin{aligned} 0 < \theta < \pi & : \alpha(\theta, \phi + 2\pi) - \alpha(\theta, \phi) = 2n\pi, \\ & n \text{ integer independent of } \theta; \\ \theta \rightarrow 0 & : \alpha(0, \phi) = a; \\ \theta \rightarrow \pi & : \alpha(\pi, \phi) = 2\phi + b; \quad a, b \text{ constants.} \end{aligned} \quad (39)$$

The θ -independence of n follows from smoothness (continuity) in θ . However these conditions on $\alpha(\theta, \phi)$ are mutually inconsistent:

$$\begin{aligned} \alpha(0, \phi) = a & \Rightarrow n = 0; \\ \alpha(\pi, \phi) = 2\phi + b & \Rightarrow n = 2. \end{aligned} \quad (40)$$

This is therefore a simple analytic proof of the nonparalelizability of TS_{dir}^2 , as mentioned earlier.

We now return to the problem of constructing a group element $u(\theta, \phi) \in SU(2)$ obeying the conditions (38). For this we exploit the construction of $u_0(\theta, \phi)$ in Eqn. (13), with behaviours as $\theta \rightarrow 0, \pi$ as given in Eqns. (15,16). Apart from a scale change in ϕ , these are qualitatively similar to the desired behaviours of $u(\theta, \phi)$. If we note that

$$\begin{aligned} \tau_3 e^{i\phi\tau_1} &= e^{-i\phi\tau_1} \tau_3, \\ e^{-i\phi\tau_1} &= e^{-i\frac{\pi}{4}\tau_2} e^{-i\phi\tau_3} e^{i\frac{\pi}{4}\tau_2}, \end{aligned} \quad (41)$$

we can see that a possible solution to our problem is

$$u(\theta, \phi) = e^{i\frac{\pi}{4}\tau_2} S_0^{-1} u_0(\theta, 2\phi) e^{-i\frac{\pi}{4}\tau_2}. \quad (42)$$

As long as θ remains in the range $0 \leq \theta \leq \pi$, for $0 < \theta < \pi$ the 2π periodicity of $u(\theta, \phi)$ in ϕ is satisfied. At the poles we find:

$$\begin{aligned} u(\theta, \phi) &\xrightarrow{\theta \rightarrow 0} \mathbb{1}, \quad \text{i.e., } u_0 = \mathbb{1}; \\ u(\theta, \phi) &\xrightarrow{\theta \rightarrow \pi} e^{-2i\phi\tau_3} \cdot \tau_2 \tau_3, \quad \text{i.e., } u_\pi = i\tau_1. \end{aligned} \quad (43)$$

Thus $u(\theta, \phi)$ is indeed a smooth function of $(\theta, \phi) \in S_{\text{dir}}^2$ at all points except the South pole, with the desired multivaluedness at that pole. The scale change $\phi \rightarrow 2\phi$ involved in Eqn. (42) does not cause any difficulties. We now use Eqn. (42) in Eqn. (36) to get a globally smooth $SU(3)$ element $\mathcal{A}_0(\theta, \phi)$ obeying Eqns. (28). After some

simplifications using the covariance condition (7) we arrive at the expressions:

$$\begin{aligned} u_0(\theta, \phi) &= S_0 e^{-i\frac{\phi}{2}\tau_1} e^{-i\frac{\phi}{2}\tau_3} e^{i\frac{\phi}{2}\tau_1}; \\ u(\theta, \phi) &= e^{-i\phi\tau_3} e^{i\frac{\phi}{2}\tau_1} e^{i\phi\tau_3}; \\ \mathcal{A}_0(\theta, \phi) &= R_0(\theta, \phi) \begin{pmatrix} u(\theta, \phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (44)$$

At the poles, $\mathcal{A}_0(\theta, \phi)$ smoothly approaches the unambiguous limiting values

$$\mathcal{A}_0(0, \phi) = \mathbb{1}, \quad \mathcal{A}_0(\pi, \phi) = \text{diag}(-i, -i, -1). \quad (45)$$

Thus according to Eqn. (28), at the poles we have the orthonormal bases for the complexified tangent spaces given by

$$\mathbf{E}^{(a)}(0, \phi) = \mathbf{e}_a, \quad \mathbf{E}^{(a)}(\pi, \phi) = -i\mathbf{e}_a, \quad a = 1, 2. \quad (46)$$

All of these correspond to linear polarizations. On the other hand, at points along the equator $\theta = \pi/2$, we have ($S = \sin \phi, C = \cos \phi, 0 \leq \phi < 2\pi$):

$$\begin{aligned} R_0\left(\frac{\pi}{2}, \phi\right) &= \begin{pmatrix} S^2 & -SC & C \\ -SC & C^2 & S \\ -C & -S & 0 \end{pmatrix}; \\ u\left(\frac{\pi}{2}, \phi\right) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i(C^2 - S^2) & 2iSC \\ 2iSC & 1 - i(C^2 - S^2) \end{pmatrix}; \\ \mathcal{A}_0\left(\frac{\pi}{2}, \phi\right) &= e^{-i\pi/4} \begin{pmatrix} S^2 & -SC & e^{i\pi/4}C \\ -SC & C^2 & e^{i\pi/4}S \\ -iC & -iS & 0 \end{pmatrix}. \end{aligned} \quad (47)$$

The complex orthonormal basis vectors for the complexified tangent space to S_{dir}^2 at the equatorial point $(\pi/2, \phi)$ are thus

$$\begin{aligned} \mathbf{E}^{(1)}(\pi/2, \phi) &= e^{-i\pi/4}(S^2, -SC, -iC), \\ \mathbf{E}^{(2)}(\pi/2, \phi) &= e^{-i\pi/4}(-SC, C^2, -iS), \\ S &= \sin \phi, C = \cos \phi. \end{aligned} \quad (48)$$

Each of these corresponds to a pure polarization state that changes smoothly as ϕ varies. For $\phi = 0, \pi/2, \pi, 3\pi/2$ we have linear polarizations; for $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ we have circular polarizations (of opposite senses for $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$); and for all other ϕ we have elliptic polarizations. Since we have a continuously changing propagation vector $\hat{\mathbf{k}} = (\cos \phi, \sin \phi, 0)$, it is not meaningful to display all these features on any common or fixed Poincaré sphere.

In a similar manner, the behaviours of $\mathcal{A}_0(\theta, \phi), \mathbf{E}^{(a)}(\theta, \phi)$ at other points on S_{dir}^2 can be examined, but we forego the details.

From the manner in which the $SU(2)$ elements $u(\theta, \phi)$ and the $SU(3)$ elements $\mathcal{A}_0(\theta, \phi)$ have been constructed, it would seem that the solution given above to the problem of defining globally smooth polarization bases over all of S_{dir}^2 is in some sense both minimal and natural.

The most general globally smooth $\mathcal{A}(\theta, \phi)$ is clearly of the form

$$\mathcal{A}(\theta, \phi) = \mathcal{A}_0(\theta, \phi) \begin{pmatrix} u'(\theta, \phi) & 0 \\ 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (49)$$

where $u'(\theta, \phi)$ is any globally smooth map $\mathbb{S}^2 \rightarrow SU(2) \simeq \mathbb{S}^3$. Any number of examples of such $u'(\theta, \phi)$ are easily constructed; for instance

$$u'(\theta, \phi) = \exp(i \sin \theta \mathbf{a}(\theta, \cos \phi, \sin \phi) \cdot \boldsymbol{\tau}) \quad (50)$$

for any real $\mathbf{a}(\theta, \cos \phi, \sin \phi)$ smooth in θ and polynomial in $\cos \phi$ and $\sin \phi$ is acceptable.

It is a result of homotopy theory that any two globally smooth choices of $u'(\theta, \phi)$ can be continuously deformed into one another; the reason is that the homotopy group $\pi_2(\mathbb{S}^3)$ is trivial [9]. Thus we see that while there are infinitely many choices of globally smooth complex orthonormal bases $\{\mathbf{E}^{(a)}(\hat{\mathbf{k}})\}$ for electric field vectors for plane waves for all $\hat{\mathbf{k}} \in \mathbb{S}_{\text{dir}}^2$, any two choices can be smoothly deformed into one another. In this precise sense, the solution given above using $\mathcal{A}_0(\theta, \phi)$ is essentially unique; any other solution is in the same homotopy class as this one, indeed there is only one homotopy class.

IV. CONCLUDING REMARKS

We have looked at three physically important and interesting problems in classical polarization optics, all of which have a global character. Our principal results, can be summarised as follows:

- [A] The first problem regarding the Poincaré sphere was treated in [3]. It was shown there that there is no globally smooth way of defining electric field vectors covering the entire Poincaré sphere. Of course, we cannot change this result, but we give a much simpler proof than that in [3], using $SU(2)$ coset space properties. We also construct an almost globally smooth coset representative, good everywhere

except at South pole, which is used later for a different problem.

- [B] As to the second problem, here again we have the well known result : no globally smooth choice of real tangent vectors is possible over the entire sphere of directions. Here we give a simple proof using $SO(3)$ coset space, as well as a simple analytic proof. Admittedly, the negative result remains unchanged but we go a step further in that we explicitly construct an almost globally smooth $SO(3)$ coset representative, problem only at South pole, for use in the next problem.

- [C] If in B above one goes from real to complex tangent vectors, it was shown in [4] that globally smooth bases do exist. We give an explicit construction of such a basis by combining the almost globally smooth expressions in A and B, using the properties of $SU(3)$, to get a truly globally smooth solution for problem C. The South pole problems in A and B can be made to annihilate one another! The novelty is in use of $SU(3)$, and the last result is that any two global choices can be smoothly deformed into one another; in that sense our group theory based solution is essentially unique.

The fact that the three-dimensional groups $SU(2)$ and $SO(3)$ play important roles in these problems comes as no surprise. On the other hand, the use of the eight-dimensional group $SU(3)$, in a limited way, in answering the third question is particularly interesting. It is likely that such approaches will be found useful in other problems in the polarization optics context as well.

It is likely that such approaches will be found useful in other problems in polarization optics such as in the Poincaré sphere based descriptions and its generalizations to situations which involve orbital angular momentum and higher order effects [10–13].

ACKNOWLEDGMENTS

Arvind acknowledges funding from DST India under Grant No. EMR/2014/000297. NM thanks the Indian National Science Academy for enabling this work through the INSA C V Raman Research Professorship.

-
- [1] H. Poincare, *Theorie Mathematique de la Lumiere* (Gauthiers-Villars, Paris, France, 1892).
[2] M. Born and E. Wolf, *Principles of Optics, 7th Ed.* (Cambridge University Press, Cambridge, England, 1999).
[3] R. Nityananda, Pramana J. Phys. **12**, 257 (1979).
[4] R. Nityananda and S. Sridhar, Annals Phys. **341**, 117 (2014).
[5] N. Mukunda, S. Chaturvedi, and R. Simon, J. Opt. Soc. Amer. A: Opt. Image Sci. Vision **31**, 1141 (2014).
[6] N. Mukunda, Arvind, S. Chaturvedi, and R. Simon, Phys. Rev. A **65**, 121021 (2002).
[7] W. Chinn and N. Steenrod, *First Concepts of Topology* (Mathematical Association of America, Washington, DC, 1966).
[8] M. Eisenberg and R. Guy, Amer. Math. Monthly **86**, 571574 (1979).
[9] M. Nakahara, *Geometry and Topology and Physics* (IOP, Bristol and Philadelphia, 2003).

- [10] M. J. Padgett and J. Courtial, Optics Lett. **24**, 430 (1999).
- [11] G. S. Agarwal, J. Opt. Soc. Amer. A: Opt. Image Sci. Vision **16**, 2914 (1999).
- [12] G. Milione, H. Sztul, D. Nolan, and R. Alfano, Phys. Rev. Lett. **107**, 053601 (2011).
- [13] M. R. Dennis and M. Alonso, Phil. T. Roy. Soc. A **375**, 20150441 (2017).